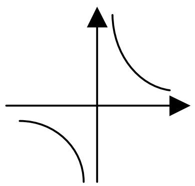
ASYMPTOTIC CURVES

The graph of $y = x^2$ is valid for all values of x

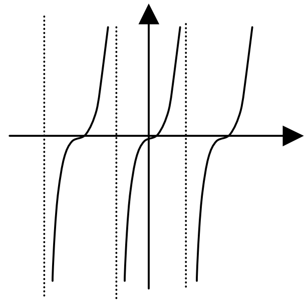
The graph of $y = \tan x$ has many vertical asymptotes.



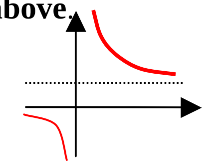
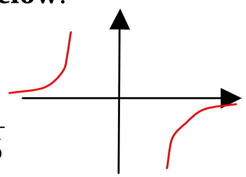
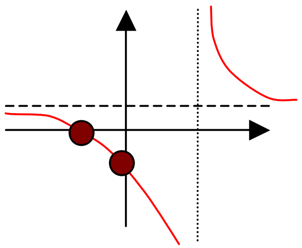
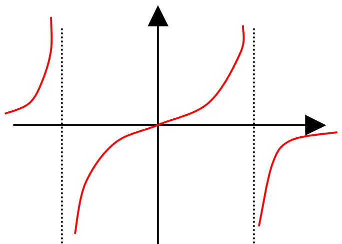
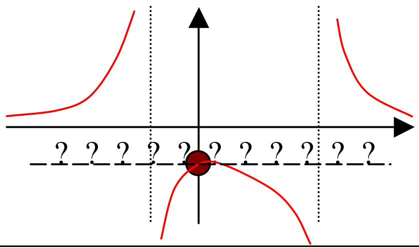
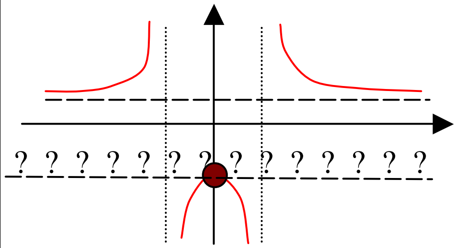
This is the basic curve $y = \frac{1}{x}$ and has a denominator x.

The axes are **asymptotes** to the curve. The curve touches the axes at infinity.

We will look at curves, which are based on the curve $y = \frac{1}{x}$ and have linear and quadratic functions of x in the denominator.



CURVE	$y = \frac{x+2}{x-2}$	$y = \frac{x}{4-x^2}$	$y = \frac{1}{(x+1)(x-2)}$	$y = \frac{x^2+9}{x^2-1}$
DESCRIPTION	Linear denominator	Unfactorised quadratic denominator	Factorised quadratic denominator.	Unfactorised quadratic denominator
1. Put x = 0 to find the intercept on the y - axis	$y = -2$	$y = 0$	$y = -\frac{1}{2}$	$y = -9$
2. Put y = 0 to find the intercept on the x – axis. A fraction will be zero when the **numerator** is zero.	$0 = x + 2$ $x = -2$	$x = 0$ This curve passes through (0, 0)	y can **never** be zero. This curve never crosses the x-axis.	$x^2 + 9 = 0$ $x^2 = -9$ No solutions so the curve never crosses the x-axis.
A fraction will be infinity when the **denominator** is zero. 3. Put the denominator equal to zero.	$0 = x - 2$ $x = 2$ This gives us our first equation of an asymptote	$4 - x^2 = 0$ $(4 + x)(4 - x) = 0$ $x = -4, \quad x = +4$ There are two vertical asymptotes here!	$(x + 1)(x – 2) = 0$ $x = -1, \quad x = 2$ There are two vertical asymptotes here as well!	$(x + 1)(x – 1) = 0$ $x = -1, \quad x = 1$ There are two vertical asymptotes here as well!

4. Consider the curve for large positive values of x ---- say 100. And large negative values. Write x ⟶ ∞ to mean "as x gets bigger and bigger".	As x ⟶ ∞ y ⟶ 1 At x = 100 y = $\frac{102}{98}$ Which is just over 1. So y gets closer to 1 from **above**. $\frac{-98}{-102}$ 	As x ⟶ ∞ y ⟶ 0 At x = 100 y = $\frac{100}{-9996}$ Which gets closer to 0 from **below**. $\frac{-100}{-9996}$ 	y = $\frac{1}{x^2-x-2}$ As x ⟶ ∞ y ⟶ 0 At x = 100 y = $\frac{1}{something\ large}$ Positive, so from above. If x is large and negative y is also positive because of the x^2.	As x ⟶ ∞ y ⟶ 1 At x = 100 y = $\frac{10009}{9999}$ just over 1 so approaching the line y = 1 from above. Exactly the same for x large and negative because of the x^2.
5. Substitute x = -x and if f(x) = f(-x) we know we have an even function which is symmetrical about the y-axis	$\frac{x+2}{x-2} \neq \frac{-x+2}{-x-2}$ Not symmetrical about the y-axis.	$\frac{x}{4-x^2} \neq \frac{-x}{4-x^2}$ Not symmetrical about the y-axis.	$\frac{1}{x^2-x-2} \neq \frac{1}{x^2+x-2}$ Not symmetrical about the y-axis.	$\frac{x^2+9}{x^2-1} = \frac{x^2+9}{(-x)^2-1}$ This curve is symmetrical about the y-axis.
6. Draw the axes. Put in asymptotes. **Sketch the curve.**				
Alternative methods of sketching.	y = $\frac{x+2}{x-2}$ may be rewritten y = $\frac{x-2+2+2}{x-2}$ y = $1+\frac{4}{x-2}$ y = $\frac{1}{x}$ Which is the graph of transformed 2 units to the left (x-2) and 1 unit up.			

7. The forbidden zone. Sometimes there is a narrow horizontal band which the curve never enters.	Here there is only the horizontal line y = 1 which the curve never crosses.	Here the curve never crosses the x-axis.	What happens between the lines $y = 0$ and $y = -\frac{1}{2}$? Does it go higher than $y = -\frac{1}{2}$	What happens between the lines $y = 1$ and $y = -9$? Does it go higher than $y = -9$?

We can see that the curve $y = \dfrac{1}{(x+1)(x-2)}$ crosses the lines y=1, y=2, etc. even y = 2.5

And if we solve the curve with the line y = 2.5 we would get a quadratic equation with the roots being the x values where the curve crosses the line.

This would also work if we take any line so let us take the line y = k.

$k = \dfrac{1}{(x+1)(x-2)} \quad \Rightarrow \quad k(x^2 - x - 2) = 1 \quad \Rightarrow \quad kx^2 - kx - (2k+1) = 0$ **eqn 1**

For solutions to exist "$b^2 - 4ac$" $\geq 0 \quad \Rightarrow \quad k^2 + 4k(2k+1) \geq 0 \quad \Rightarrow \quad 9k^2 + 4k \geq 0$

Solving this quadratic inequality in k, gives $k \geq 0$, $k \leq -\frac{4}{9}$

The forbidden zone: The curve will not cross the line $y = k$ if $-\frac{4}{9} < k < 0$

If we put $k = -\frac{4}{9}$ into **eqn 1**, we get the repeated roots $x = 2$. So $(2, -\frac{4}{9})$ is the maximum point.

$k = \dfrac{x^2+9}{x^2-1} \quad \Rightarrow \quad k(x^2 - 1) = (x^2 + 9)$

$\Rightarrow \quad (k-1)x^2 - (k+9) = 0$ **eqn 1**

For solutions to exist "$b^2 - 4ac$" ≥ 0

$\Rightarrow \quad 0^2 + 4(k-1)(k+9) \geq 0$

with solutions $k \geq 1$, $k \leq -9$

The curve will not cross the line $y = k$ if $-9 < k < 1$

If we put $k = -9$ into **eqn 1**, we get the repeated roots $x = 0$. So $(0, -9)$ is the maximum point.

Finally, make sure that there is only one value of y corresponding to any value of x. I.e. the curve is not a one-to-many function.