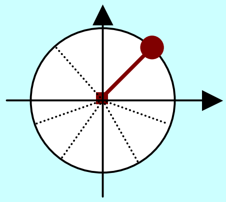
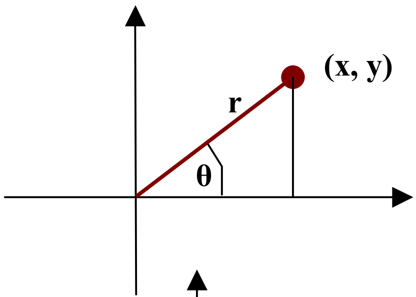
Equations of circles



The locus of all points, which are a certain distance from a fixed point is a circle.

As the radius rotates like a 'radar screen', the path of the tip of the line describes a circle.

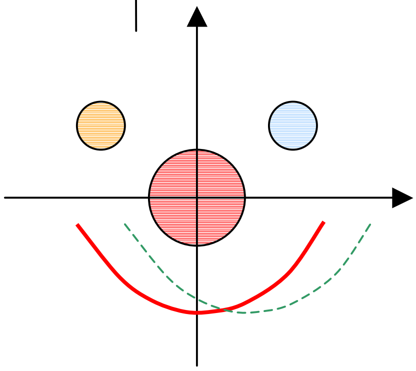


Any point (x, y) on the circle may be written in terms of the angle θ, which the radius (**r**) makes with the x-axis.

$x = r\cos\theta$ $y = r\sin\theta$ using Pythagoras on the triangle.

These two equations are the parametric equations of the circle with radius r and centre (0, 0).

Eliminating θ will give us the Cartesian equation of this circle.
Square and add the two equations:
$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta \;=\; r^2(\cos^2\theta + \sin^2\theta)$
since $\cos^2\theta + \sin^2\theta = 1$, **the Cartesian equation of the circle centre (0, 0) and radius r is** $\underline{x^2 + y^2 = r^2}$



The equation of the clown's nose is **$x^2 + y^2 = 9$** (radius = 3)
The smile is a basic parabola **y = f(x),** which has been **transformed** 7 units **down**. Its equation is y = f(x) – 7 which can also be written

y+ 7 = f(x)

Transformations Now, moving the smile 3 units to the right changes the equation to **y + 7 = f(x – 3).**
If the nose circle had radius = 2 its equation would be $x^2 + y^2 = 4$
If we transform this circle 6 units to the right and 5 units up its equation would be **$(x-6)^2 + (y-5)^2 = 4$** **the right eye.**

The equation of the left eye is $(x+6)^2 + (y-5)^2 = 4$

See the factsheet on **functions** for notes on transformations.

The general equation of a circle With centre (a, b) and radius r is

$\underline{(x-a)^2 + (y-b)^2 = r^2}$ **#1**

Other forms of the equation:
$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$.
$x^2 + y^2 - 2ax - 2by = r^2 - a^2 - b^2$

Finding the centre and radius.

A circle with centre C has equation **$x^2 + y^2 - 4x + 6y = 12$**. We need to get this into the form **#1** to determine the centre and the radius. Rearrange the equation and use the method of completing the square.

$x^2 - 4x + y^2 + 6y = 12$: $(x-2)^2 - 4 + (y+3)^2 - 9 = 12$ → **$(x-2)^2 + (y+3)^2 = 25$**
The centre is (2, -3) and the radius is 5.

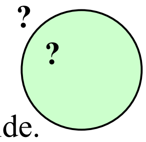
Verifying a point lies on a circle and find the equation of the tangent at that point

To verify that the point (5, 1) lies on the circle **$x^2 + y^2 - 4x + 6y = 12$,** simply substitute the values x = 5, y = 1 into the equation of the curve and check they fit the equation.

Left hand side: $5^2 + 1^2 - 4(5) + 6(1) = 25 + 1 - 20 + 6 = 12$ Right hand side.

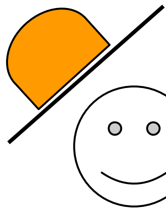
To find the equation of the tangent we could differentiate implicitly and find the gradient………Try it!
We could look at the centre O(2,-3) and the point on the circumference P(5,1) to find the gradient of OP.
Gradient = 4/3 and the tangent will be perpendicular to this with gradient –3/4.
Equation of the tangent with this gradient through (5,1) is **3x + 4y = 19**

Does a point lie inside or outside a circle?



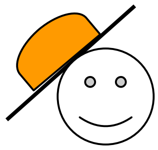
Write down the point (0, 0) or (0, 2) and the centre **(2, -3)** and apply Pythagoras
For the point (0, 0): $\sqrt{(2^2 + 3^2)} = \sqrt{13}$. Since this is **less** than the radius (5) the point is inside.
For the point (0, 2): $\sqrt{(2^2 + 5^2)} = \sqrt{29}$. Since this is **more** than the radius (5) the point is outside.

Showing that a line does not meet the circle



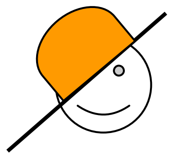
To show that the line $y = x + 3$ does not meet the circle, *pretend it does* and
Solve simultaneously with the circle $\mathbf{x^2 + y^2 - 4x + 6y = 12}$
$\mathbf{x^2 + (x+3)^2 - 4x + 6(x+3) = 12}$ **leads to the quadratic** $\mathbf{2x^2 + 8x + 15 = 0}$
Investigation of **"$b^2 - 4ac$"** shows it to be $64 - 4(2)(15) = -56$.
Since **$b^2 - 4ac < 0$ there are no real roots.**

Showing that a line touches the circle



To show that the line $y = 2$ touches the circle:
Solve simultaneously with the circle $\mathbf{x^2 + y^2 - 4x + 6y = 12}$
$\mathbf{x^2 + (2)^2 - 4x + 6(2) = 12}$ **leads to the quadratic** $\mathbf{x^2 - 4x + 4 = 0}$
Since **"$b^2 - 4ac$" = 0** **and** $\mathbf{(x - 2)(x - 2) = 0}$ **Only one solution at (2, 2).**

Showing that a line meets the circle at two distinct points



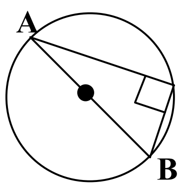
To show that the line $y = x + 2$ meets the circle at two distinct points::
Solve simultaneously with the circle $\mathbf{x^2 + y^2 - 4x + 6y = 12}$
$\mathbf{x^2 + (x+2)^2 - 4x + 6(x+2) = 12}$ **leads to the quadratic** $\mathbf{2x^2 + 6x + 4 = 0}$
i.e. $\mathbf{x^2 + 3x + 2 = 0}$ **"$b^2 - 4ac$" > 0** **and** $\mathbf{(x + 2)(x + 1) = 0}$
$x = -1$, and $x = -2$ are the solutions with $y = 1$ and $y = 0$ **The line cuts the circle at (-2, 0) and (-1, 1)**

Finding the equation of the circle with opposite ends of the diameter given.

AB is the diameter of a circle with A = (-3, 4) and B = (5, -2).

- The length of AB can be found using Pythagoras: $AB = \sqrt{(8^2 + 6^2)} = 10$. This is the diameter.
- The gradient of AB can be found if needed: grad(AB) = $-\frac{6}{8}$
- The mid point of AB is found using the mean of the coordinates: M = (1, 1). This is the centre.

So with centre (1, 1) and radius 5, the equation of the circle is $\mathbf{(x-1)^2 + (y-1)^2 = 25}$



OR with A = (x_1, y_1) and B = (x_2, y_2), any point C (x, y) on the circumference will be such that AC is perpendicular to BC since the angle in a semicircle is 90 degrees.

C **Applying Pythagoras:**

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = (x_1 - x)^2 + (y_1 - y)^2 + (x - x_2)^2 + (y - y_2)^2$$

$$x_1^2 - 2x_1x_2 + x_2^2 + y_1^2 - 2y_1y_2 + y_2^2 = x_1^2 - 2xx_1 + x^2 + y_1^2 - 2yy_1 + y^2 + x^2 - 2xx_2 + x_2^2 + y^2 - 2yy_2 + y_2^2$$

$$0 = 2x^2 + 2y^2 - 2xx_1 + 2x_1x_2 - 2yy_1 + 2y_1y_2 - 2xx_2 - 2yy_2$$

$$0 = x^2 + y^2 - xx_1 + x_1x_2 - yy_1 + y_1y_2 - xx_2 - yy_2$$

$$0 = (x - x_1)(x - x_2) + (y - y_1)(y - y_2)$$ This only needs the ends of the diameter.

If AB is the diameter of a circle with A = (-3, 4) and B = (5, -2), the equation of the circle is
$\mathbf{(x + 3)(x - 5) + (y - 4)(y + 2) = 0}$