THE EXPONENTIAL DISTRIBUTION

The number of goals scored in a five-minute period of a football match is modelled by a Poisson distribution with mean 1.

We write $\Pr(X = x) = \dfrac{e^{-2}\,2^{x}}{x!}$.

We can model a ten-minute period by a Poisson distribution with mean = 2 or a one-minute period by a Poisson distribution with mean 0.2, such is the additive quality of the Poisson when events occur at random.

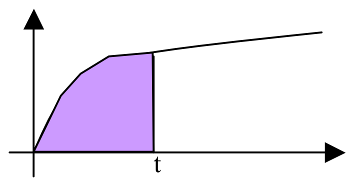
If we let time (t) vary, then the number of goals in a t-minute period is a Poisson variable with mean 0.2t. Since the number of goals per unit of time is Poisson with mean 0.2.

The probability that 5 goals will be scored in a t-minute period is $\dfrac{e^{-0.2t}\,0.2t^{5}}{5!}$

The probability that we have to wait w minutes before the first goal comes is equivalent to no goals being scored during the first w minutes. So the first sign of goal activity occurs after w minutes. The probability that the waiting time to the first goal is greater than a time ***t*** is the probability that no goals are scored in the first ***t*** minutes. $\dfrac{e^{-0.2t}\,0.2t^{0}}{0!} = e^{-0.2t}$

Therefore the probability that at least one goal is scored in the first t minutes $= 1 - e^{-0.2t}$ and this is the probability that the **waiting time is less than *t*.**

I.e. this is the cumulative distribution of the waiting time. F(t)



Waiting time to first goal

We can find the density function by differentiating:

$F'(t) = 0.2\,e^{-0.2t}$ for $t > 0$. Remember that 0.2 was the number of goals per unit time and if we denote this by λ, we can write the density function of the exponential distribution as

$f(t) = \lambda e^{-\lambda t}$, for $t > 0$. The mean is $1/\lambda$. The variance is $1/\lambda^2$.

Exponential Distributions Have No Memory

An important property of the exponential distribution is that it is "memoryless." This means that, for $0 < s < t$, $P(W > t|W > s) = P(W > t - s)$. (This is easy to show by evaluating the conditional probability on the left.)

In words, the no-memory property amounts to this: Given that there have been no counts up to time *s*, the conditional probability that there are no counts up to the later time *t* is just the probability of seeing no counts in a period of length $t - s$. The process does not "remember" that it has already been running for *s* units of time.

When applied to reliability theory, the memoryless property of the exponential distribution gives rise to the phrase "used is as good as new." If the waiting time until failure (or death) of a component is exponentially distributed, then the probability that a component that has already survived for $s = 5$ years will survive for one more year (for a total of $t = 6$ years) is the same as the probability that a brand new component will survive for $t - s = 1$ year.

The expression "used is as good as new" describes the lifetimes of certain solid-state electronic parts with an acceptable degree of realism. (They die when they have a random accident, not by wearing out. There is not much to wear out in a chunk of sand—not even in the very intricate chunks of sand we call computer chips, etc.) Unfortunately, this expression does not accurately describe the survival of humans: take $s = 70$ years and $t = 100$ years for an example. (Humans can die by accident, but also by wearing out.)