Implicit differentiation

When finding the gradient of a curve y = f(x), we normally write $\frac{\partial y}{\partial x}$ = and differentiate with respect to x.

We could also write $\frac{\partial}{\partial x}$f(x) to show that we are differentiating the function with respect to x.

This is fine as long as we can write the equation of the curve in the form y = f(x).

For instance, the gradient of the circle $x^2 + y^2 = 16$ may be found by writing $y = \sqrt{16 - x^2}$ and using the chain rule: $\frac{\partial y}{\partial x} = -\frac{1}{2}(16 - x^2)^{-½}2x = -x(16 - x^2)^{-½}$.

Now even in a complicated function like $x^2 + y^2 + 4x + 6y = 12$, which **cannot** be written in the form y = f(x), may be differentiated.

We know y is connected to x but we can't get the connection directly.

The **implication** is there: it is **implied** so we **differentiate implicitly**.

When differentiating using function of a function or the **chain rule:** If y = f(u), where in turn u = f(x)

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \text{ x } \frac{\partial u}{\partial x}$$

So, to differentiate u^3 where $u = 2x^2 + 3x$, we write $\frac{\partial}{\partial x}(2x^2 + 3x)^3$ and get $3(2x^2 + 3x)^2(6x + 3)$.

We can do the same if we come across y^3: $\frac{\partial}{\partial x}(y^3) = 3y^2\frac{\partial y}{\partial x}$

Differentiating $x^2 + y^2 + 4x + 6y = 12$, would look like this if we differentiate across the curve:

$\frac{\partial}{\partial x}\mathbf{x^2} + \frac{\partial}{\partial x}\mathbf{y^2} + \frac{\partial}{\partial x}\mathbf{4x} + \frac{\partial}{\partial x}\mathbf{6y} = \frac{\partial}{\partial x}\mathbf{12}$, and the result would be: $2x + 2y\frac{\partial y}{\partial x} + 4 + 6\frac{\partial y}{\partial x} = 0$

The gradient can now be found when we isolate $\frac{\partial y}{\partial x}$.

$\frac{\partial y}{\partial x} = \frac{-(2x+4)}{2y+6}$ Note here that the gradient is given in terms of x **and** y.

Finding the gradient of the circle $x^2 + y^2 = 16$ using implicit differentiation

$\frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial x}y^2 = \frac{\partial}{\partial x}16 \Rightarrow 2x + 2y\frac{\partial y}{\partial x} = 0$ and $\frac{\partial y}{\partial x} = -\frac{x}{y}$

Note that this is somewhat different to $\frac{\partial y}{\partial x} = -x(16 - x^2)^{-½}$, which we obtained from rewriting the equation of the circle as $y = \sqrt{16 - x^2}$ and using the chain rule, but if we substitute $y = \sqrt{16 - x^2}$ into $\frac{\partial y}{\partial x} = -\frac{x}{y}$ we get $\frac{\partial y}{\partial x} = -\frac{x}{\sqrt{16 - x^2}}$ which is identical to $-x(16 - x^2)^{-½}$.

Finding the gradient of $ax^3 + by^3 + cx^2 + dy^2 + ex + fy + gxy + hx^2y + j\frac{x}{y} = 5$

Differentiate with respect to x (w.r.t x) and use the table below as you come across each element of the curve.

ELEMENT	DIFFERENTIAL	COMMENTS
x^n	nx^{n-1}	Straight differential w.r.t. x
y^n	$ny^{n-1}\frac{\partial y}{\partial x}$	Using the chain rule for implicit y.
$x^n y^n$	$x^n ny^{n-1}\frac{\partial y}{\partial x} + y^n nx^{n-1}$	Using the product rule with implicit y *Write down the first, differentiate the second* ***plus*** *write down the second, differentiate the first.*
$\frac{x}{y}$	$\dfrac{y1 - x\frac{\partial y}{\partial x}}{y^2}$	Using the quotient rule with implicit y *Write down the bottom, differentiate the top* ***minus*** *write down the top, differentiate the bottom All over the bottom squared.*

$$\frac{\partial}{\partial x}(\mathbf{a}x^3 + \mathbf{b}y^3 + \mathbf{c}x^2 + \mathbf{d}y^2 + \mathbf{e}x + \mathbf{f}y + \mathbf{g}xy + \mathbf{h}x^2y + \mathbf{j}\frac{x}{y})$$

where a, b, c, d, e, f, g, h and j are all constants.

$$= 3\mathbf{a}x^2 + 3\mathbf{b}y^2\frac{\partial y}{\partial x} + 2\mathbf{c}x + 2\mathbf{d}y\frac{\partial y}{\partial x} + \mathbf{e} + \mathbf{f}\frac{\partial y}{\partial x} + \mathbf{g}(x\frac{\partial y}{\partial x} + y1) + \mathbf{h}(x^2\frac{\partial y}{\partial x} + y2x) + \mathbf{j}\frac{(y1 - x\frac{\partial y}{\partial x})}{y^2}$$

The chain rule.

Simply put: $\frac{\partial}{\partial x}x^n = nx^{n-1}$ but if we have to differentiate a function of x which is in the form $\{f(x)\}^n$ then the differential becomes $n\{f(x)\}^{n-1} \times f'(x)$ where we have to "ignore" the function of x and then multiply by its differential. $f'(x)$ is the differential of $f(x)$.

If $y = f(u)$, where $u = f(x)$ then $\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \times \frac{\partial u}{\partial x}$

$\frac{\partial}{\partial x}e^x = e^x$ so $\frac{\partial}{\partial x}e^{f(x)} = e^{f(x)} \times f'(x)$ $\qquad \frac{\partial}{\partial x}e^{3x+2} \longrightarrow e^{3x+2} \times 3$

$\frac{\partial}{\partial x}\ln x = \frac{1}{x}$ so $\frac{\partial}{\partial x}\ln f(x) = \frac{1}{f(x)} \times f'(x)$ $\qquad \frac{\partial}{\partial x}\ln(3x^2 + 4x) \longrightarrow \frac{1}{3x^2 + 4x} \times (6x+4)$

$\frac{\partial}{\partial x}\sin x = \cos x$ so $\frac{\partial}{\partial x}\sin f(x) = \cos f(x) \times f'(x)$ $\qquad \frac{\partial}{\partial x}\sin(3x^2 + 4x) \longrightarrow \cos(3x^2 + 4x) \times (6x+4)$