∫ntegration (p3)

Integration by Inspection

When differentiating using function of a function or the **chain rule:** If y = f(u), where in turn u = f(x)

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \times \frac{\partial u}{\partial x}$$

So, to differentiate u^3 where $u = x^2 + 3$, we write $\frac{\partial}{\partial x}(x^2 + 3)^3$ and get $3(x^2 + 3)^2(2x)$.

The result is a **product** showing a **main function** and its **derivative** or almost its derivative.

An integral of the form $\int (x^2 + 3)^2 x$ contains a main function and almost its derivative and to find the integral we differentiate the function which leads to the main function.

Differentiating $(x^2 + 3)^3$ gives $6x(x^2 + 3)^2$ which is the function $\left\{ \int (x^2 + 3)^2 x. \right\}$ we are trying to integrate but with an extra factor of 6.

Differentiating $(x^2 + 3)^3$ **divided by 6** would give us the exact function we wish to integrate,

$$\frac{\partial}{\partial x}\frac{(x^2+3)^3}{6} \longrightarrow \frac{6x(x^2+3)^2}{6} \qquad \textbf{so} \int \frac{6x(x^2+3)^2}{6} \longrightarrow \frac{(x^2+3)^3}{6}$$

We have **compensated** by dividing by 6 and simply reversed the process to get the integral.

Question	Differentiate and compensate to get the exact integral		Compensate by:	Answer
$\int (x^3 + 3)^4 x^2\,dx$	$\frac{\partial}{\partial x}\frac{(x^3+3)^5}{15}$	$\longrightarrow \frac{5(x^3+3)^4 3x^2}{15}$	**Dividing by 15**	$\frac{(x^3+3)^5}{15} + c$
$\int x\sqrt{(x^2+2)}\,dx$	$\frac{\partial}{\partial x}\frac{(x^2+2)^{\frac{3}{2}}}{3}$	$\longrightarrow \frac{3(x^2+2)^{\frac{1}{2}}2x}{2*3}$	**Dividing by 3**	$\frac{(x^2+2)^{\frac{3}{2}}}{3} + c$
$\int \sin^3 x \cos x\,dx$	$\frac{\partial}{\partial x}\frac{\sin^4 x}{4}$	$\longrightarrow \frac{4\sin^3 x\cos x}{4}$	**Dividing by 4**	$\frac{\sin^4 x}{4} + c$
$\int \frac{xdx}{(x^2+1)^2}$	$\frac{\partial}{\partial x}\frac{-(x^2+1)^{-1}}{2}$	$\longrightarrow \frac{(x^2+1)^{-2}2x}{2}$	**Multiplying by –1 and dividing by 2**	$\frac{-(x^2+1)^{-1}}{2} + c$

Simple standard integrals to learn.

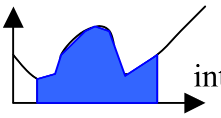
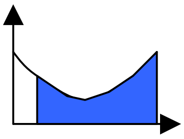
Since $\frac{\partial}{\partial x}\sin ax = a\cos ax$ and $\frac{\partial}{\partial x}\frac{\sin ax}{a} = \frac{a\cos ax}{a}$, $\qquad \int \cos ax\,dx = \frac{\sin ax}{a} + c$

Since $\frac{\partial}{\partial x}\cos ax = -a\sin ax$ and $\frac{\partial}{\partial x}\frac{-\cos ax}{a} = \frac{a\sin ax}{a}$, $\qquad \int \sin ax\,dx = -\frac{\cos ax}{a} + c$

Since $\frac{\partial}{\partial x}e^{ax} = a\,e^{ax}$ and $\frac{\partial}{\partial x}\frac{e^{ax}}{a} = \frac{ae^{ax}}{a}$, $\qquad \int e^{ax}\,dx = \frac{e^{ax}}{a} + c$

All the above have been done by inspection but there are some integrals which may have to be done by substitution.

Integration by Substitution

The idea is to turn a difficult integral in terms of x:  into an easier one in terms of u 

Consider $\int (\mathbf{x^3 + 3})^4 x^2\, dx$ Eliminate the main obstacle by letting u = $x^3 + 3$

Differentiate to get the link between dx and du: $\frac{\partial u}{\partial x} = 3x^2$ therefore $dx = \frac{\partial u}{3x^2}$

Now substitute to completely eliminate all traces of x: $\int u^4 \frac{x^2 du}{3x^2} = \int \frac{u^4 du}{3} = \frac{u^5}{15}$

To get the answer back in terms of the original x substitute back for u: $\frac{(x^3+3)^5}{15}$

If there are limits then change them in accordance with the substitution: $u = x^3 + 3$

$\int_0^1 (x^3+3)^4 x^2 dx$ becomes $\int_3^4 \frac{u^4 du}{3} = \left[\frac{u^5}{15}\right]_{x=3}^{x=4} = = \left[\frac{781}{15}\right]$.

Now consider $\int \frac{xdx}{(x+1)^2}$ This cannot be done by inspection. We cannot compensate by dividing by the variable x. This **has** to be done by substitution. Let u = x + 1 $\frac{\partial u}{\partial x} = 1$, du = dx.

The integral becomes $\int \frac{(u-1)du}{(u)^2} = \int \frac{1}{u} - \frac{1}{u^2} du$ = ln u - $\frac{1}{u}$ + c

Now consider $\int \frac{dx}{1+x^2}$ The substitution $u = 1 + x^2$ would not work. We have to turn to a trig substitution like this: Let x = tan θ, $\frac{\partial x}{\partial \theta} = \sec^2 \theta$, dx = $\sec^2 \theta\, d\theta$ $\theta = \tan^{-1} x$

The integral becomes $\int \frac{\sec^2 \theta\, d\theta}{1+\tan^2 \theta}$ (since $\sec^2 \theta = 1 + \tan^2 \theta$) = $\int d\theta = \theta = \tan^{-1} x + c$

Logarithmic Integration

Applying the chain rule to ln f(x): $\frac{\partial}{\partial x} \ln f(x) = \frac{1}{f(x)} \times f'(x) = \frac{f'(x)}{f(x)}$ ∴ $\int \frac{f'(x)dx}{f(x)} = \ln f(x)$

The integral of a quotient, where the numerator is the differential of the denominator, is the natural log of the denominator.

$\int \frac{2xdx}{x^2+3} = \ln(x^2+3) + c$

$\int \frac{x^2+2dx}{x^2+2x-3} = \ln(x^2+2x-3) + c$

$\int \frac{x^2+1\ dx}{x^3+3x}$ here, the numerator is not the exact differential of the denominator so we have to do some compensating again. We need $3x^2 + 3$ in the numerator so put the 3 in and compensate with $\frac{1}{3}$ outside the integral sign.

$$\frac{1}{3}\int \frac{3(x^2+1)dx}{x^3+3x} \quad = \quad \frac{1}{3}\ln (x^3 + 3x) + c$$

Partial fractions leading to logarithmic integration.

Rational functions with quadratic denominators are split into partial fractions and then integrated:

$$\int \frac{4dx}{(x+1)(x-3)} \quad = \quad \int \frac{1}{(x-3)} - \frac{1}{(x+1)}dx \quad = \quad \ln (x\text{-}3) - \ln (x\text{+}1) \quad = \quad \ln \frac{(x-3)}{(x+1)} + c$$

Integration by parts - used with mixed products

When differentiating a product: $\frac{\partial}{\partial x}uv \; = \; u\frac{\partial v}{\partial x} \; + \; v\frac{\partial u}{\partial x}$ and the reverse: $\int u\frac{dv}{dx} + v\frac{du}{dx} \; = \; uv$

This can be written $\int u\frac{dv}{dx} + \int v\frac{du}{dx} \; = \; uv$ **or**

$$\int u\frac{dv}{dx} \;\; = \;\; \mathbf{uv} \; - \; \int v\frac{du}{dx}$$

$\int xe^x dx$ designate u so that when it is differentiated it becomes simpler: Let u = x, $\frac{\partial v}{dx} = e^x$

$$\int xe^x dx \; = \; xe^x \; - \; \int e^x.1dx \; = \; xe^x \; - \; e^x \; + \; c$$

$$\int x\cos x dx \; = \; x \sin x \; - \; \int \sin x.1dx \; = \; x \sin x \; + \; \cos x + \; c$$

u and $\frac{\partial v}{dx}$ are chosen and the answer found by taking out uv and then performing a second simpler integration.

Sometimes after "squeezing out" the second integral needs Further squeezing out and leaves a third integral.

$$\int x^2 \sin x dx \; = \; x^2 (\text{-}\cos x) \; - \; \int (-\cos x)2x dx$$

u $\frac{\partial v}{dx}$ u v - v $\frac{\partial u}{dx}$

$$= \; -x^2 \cos x \; + \; \int \cos x 2x dx \; = \; -x^2 \cos x \; + \; 2x \sin x \; - \; \int \sin x \cdot 2 \cdot dx$$

$\frac{\partial v}{dx}$ u u v v $\frac{\partial u}{dx}$

$$= \; -x^2 \cos x \; + \; 2x \sin x \; + \; 2 \cos x \; + \; \mathbf{c}$$

Integration by parts is used to integrate the function ln x by writing 1.ln x and letting u = ln x

$$\int 1.\ln x dx \; = \; \ln x \,.\, x \; - \; \int x\frac{1}{x}dx \; = \; x \ln x \; - \; x \; + \; c$$

Further integration methods involve using the **trigonometric identities.**

Integration of odd powers of sin x and cos x

When differentiating $\sin^n x$ and $\cos^n x$ we get $n \sin^{n-1}x \cos x$ and $n \cos^{n-1}x \sin x$ respectively
There is always a single sinx or cosx tagged on so we prepare for this when integrating odd powers.

$\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx$ Now use the identity $\mathbf{\cos^2 x + \sin^2 x = 1}$

$\int (1 - \cos^2 x) \sin x\, dx = \int \sin x - \cos^2 x \sin x\, dx = -\cos x + \dfrac{\cos^3 x}{3} + c$

since $\dfrac{\partial}{\partial x}\cos^3 x = -3\cos^2 x \sin x$. Note that we compensate by dividing by 3.

Similarly $\int \cos^3 x\, dx = \int \cos^2 x \cos x\, dx = \int (1 - \sin^2 x) \cos x\, dx = \int \cos x - \sin^2 x \cos x\, dx$

$= \sin x - \dfrac{\sin^3 x}{3} + c$

Integration of odd powers of sin x and cos x

To integrate even powers of sinx and cosx, we use the double angle formula: $\cos 2x = 1 - 2\sin^2 x$
$= 2\cos^2 x - 1$

Changing the subject of these two identities gives $\sin^2 x = \dfrac{1-\cos 2x}{2}$ and $\cos^2 x = \dfrac{\cos 2x - 1}{2}$

So $\int \sin^2 x\, dx = \int \dfrac{1-\cos 2x}{2}\, dx = \dfrac{1}{2}\left(x - \dfrac{\sin 2x}{2}\right) + c$

And $\int \cos^2 x\, dx = \int \dfrac{\cos 2x - 1}{2}\, dx = \dfrac{1}{2}\left(\dfrac{\sin 2x}{2} - x\right) + c$

Since most of the integration methods have been deduced by first considering differentiation of functions, it might be worth noting the following differentials.

$\dfrac{\partial}{\partial x}\sec^n x = n \sec^{n-1}x \sec x \tan x = n \sec^n x \tan x$ $\int \sec^n x \tan x\, dx = \dfrac{\sec^n x}{n} + c$

$\dfrac{\partial}{\partial x}\tan^n x = n \tan^{n-1}x \sec^2 x$ $\int \tan^n x \sec^2 x\, dx = \dfrac{\tan^{n+1} x}{n+1} + c$

Integration of products of sin and cos using the factor formula

$\sin A + \sin B = 2 \sin \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$ is the first of the factor formulae

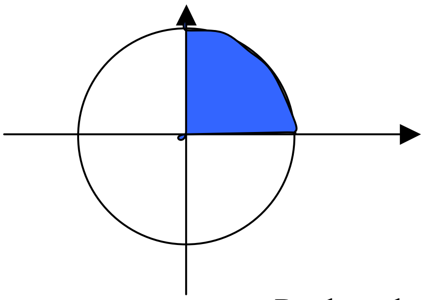
Let A = P + Q and B = P – Q, this gives the formula **sin(P+Q) + sin(P-Q) = 2 sin P cos Q** *1*
The other three formulae are: **sin(P+Q) + sin(P-Q) = 2 sin P cos Q** *2*
sin(P+Q) + sin(P-Q) = 2 sin P cos Q *3*
sin(P+Q) + sin(P-Q) = 2 sin P cos Q *4*

These formulae enable us to integrate trigonometric products like sin 5x cos x.

$\int \sin 5x \cos 3x\, dx = \int \dfrac{\sin 8x + \sin 2x}{2}$ (using equation 1 above) $= \dfrac{1}{2}\left[\dfrac{-\cos 8x}{8} + \dfrac{-\cos 2x}{2}\right] + c$

Finding the area under a curve given parametrically

The parametric equations of the circle centre (0,0) and radius r are: $x = r\cos\theta$, $y = r\sin\theta$



The area under the curve between x = 0 and x = r is written

$$\int_{x=0}^{x=r} y\,dx = \int_0^r r\sin\theta\ dx$$ There is a mixture of θ and x here so a substitution has to be made.

Replace dx by $\frac{\partial x}{\partial \theta}d\theta$ and change the limits using $\theta = \cos^{-1}\frac{x}{r}$

The integral becomes $\int_{\theta=\frac{\pi}{2}}^{\theta=0} r\sin\theta(-r\sin\theta)d\theta = -r^2\int_{\theta=\frac{\pi}{2}}^{\theta=0}\sin^2\theta\ d\theta = -r^2\left[\frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)\right]_{\frac{\pi}{2}}^{0}$

$= -r^2\left(-\frac{\pi}{4}\right) = \frac{\pi r^2}{4}$ which is a quarter of the area of a circle!

Integration checklist

1 If the integral is a product of a function and its derivative try **inspection.**
2 If the integral is a product of a function and its derivative and inspection is out, try **substitution.**
3 If the integral is a mixed product eg. (x sin x) then go for **integration by parts.**
4 If the integral is a fraction with the derivative above a function use **logarithmic integration.**
5 Quadratic denominators can be factorised and should lead to **partial fractions.**
6 For even or odd powers of sinx or cosx use the appropriate **trigonometric identities.**
7 For products of sin and cos use the **factor formula** to substitute.
8 For any other integral you are on your own!