

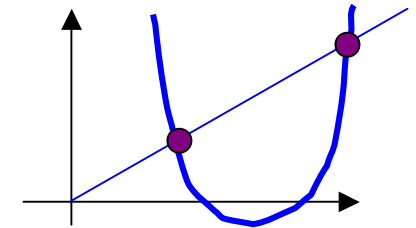
ITERATIVE METHODS

Simultaneous equations like $\begin{cases} 3x + 4y = 18 \\ 2x - y = 1 \end{cases}$
of intersection of the lines $3x + 4y = 18$ and $2x - y = 1$.

have a solution $x = 2, y = 3$ which can be viewed as the point

The intersection of the curve $y = x^2 - 5x + 6$ and the line $y = 0$ leads to the quadratic equation $x^2 - 5x + 6 = 0$
This quadratic equation gives solutions $x = 2, x = 3$.

The intersections of the curve $y = x^2 - 5x + 6$ and the line $y = x$ can be seen on the diagram



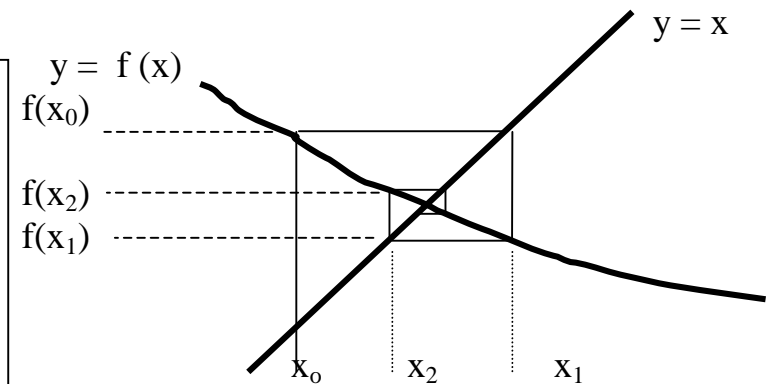
The solution is found by solving the two equations simultaneously: $x^2 - 5x + 6 = x$
Giving the quadratic equation: $x^2 - 6x + 6 = 0$
Solutions are: (using the quadratic formula) 1.27 and 4.73

The solutions could also have been found from plotting the graphs and reading off or from trial and improvement.

Sometimes there is no formula so an iterative method is used. This is similar to trial and improvement.

Our iteration involves rearranging the equation into the form $f(x) = x$ and taking a starting value near the solution.

If the start value x_0 is substituted into $f(x)$ we get the value $f(x_0)$
If this y value is taken back to meet the line $y = x$ and then down to meet the x axis we will have a value x_1 which we can substitute into $f(x)$ to get $f(x_1)$
This value is now taken as x_2 because the line $y = x$ has all y values equal to x values.
If we continue in this manner we get better and better values for the solution of $f(x) = x$. This method of getting nearer and nearer the solution looks like a spider's web and is known as ITERATION.



EXAMPLE

Solve the equation $e^x = 3 - x$. We first need to get the equation into the form $f(x) = x$

Taking logs gives $x = \ln(3 - x)$ and we are ready to start using the iterative formula $x_{n+1} = \ln(3 - x_n)$

A sketch might help us to get the start value but the question may help:

Take $x_0 = 0.8$	$x_0 = 0.8$	$x_1 = \ln(3 - 0.8) = 0.7885$
	$x_1 = 0.7885$	$x_2 = \ln(3 - 0.7885) = 0.7937$
	$x_2 = 0.7937$	$x_3 = \ln(3 - 0.7937) = 0.7913$
	$x_2 = 0.7913$	$x_3 = \ln(3 - 0.7913) = 0.7924$
	$x_3 = 0.7924$	$x_1 = \ln(3 - 0.7924) = 0.7919$

The values in the right hand column are closing in to 0.79 to 2 dec. pl. **The solution is 0.79 (2dp).**

You may have noticed that a simpler way to get the original equation into the form $f(x) = x$ is to write $x = 3 - e^x$

However, if we start with substituting $x = 0.8$ into $3 - e^x$ and continue with subsequent substitutions we get:

0.7745, 0.8306, 0.7053, 0.9755, 0.3475,This is not closing in to an answer - **it is not converging**

We say that it is **diverging**.

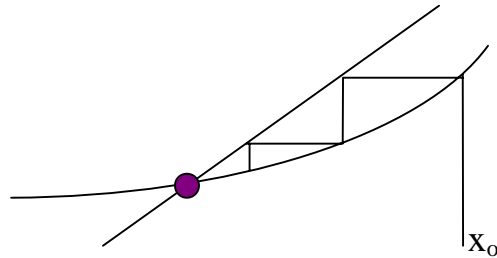
There are different ways of rearranging an equation into the form $f(x) = x$ but they all won't converge.

Success in convergence will depend on the gradient of $f(x)$ at the point of intersection of $f(x)$ and the line $y = x$.

If the gradient at the point of intersection is:

Between 0 and 1

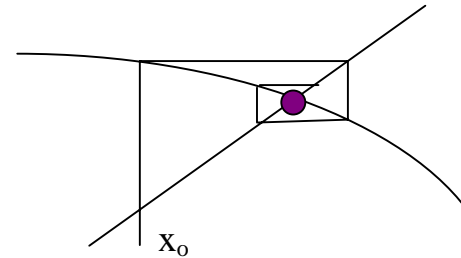
$$0 < \text{grad} < 1$$



The sequence $x_0 x_1 x_2 x_3$
moves toward the root from one direction

between -1 and 0

$$-1 < \text{grad} < 0$$



The sequence $x_0 x_1 x_2 x_3$
moves toward the root from alternate directions

If the gradient at the point of intersection is greater than 1 or less than -1 the sequence will diverge.

Note: Don't forget to use radians if trigonometric functions are involved.

Example

Solve $x = \frac{\sin x}{x}$ with $x_0 = 0.5$

Solution: 0.5, 0.9589, 0.8536, 0.8829, 0.8751, 0.8771, 0.8767.....

The root is 0.88 to 2 decimal places.