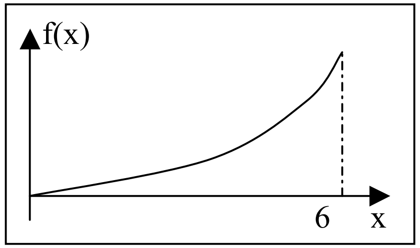
# CONTINUOUS PROBABILITY DISTRIBUTIONS

The length of worms in my garden can be modelled by a continuous probability distribution. As it is a probability distribution, I need to supply the **range** of possibilities together with the probabilities of occurrence.
For a continuous distribution this is expressed as a function of x: $f(x) = kx^2$
With probabilities over the range: $0 \le x \le 6$, no other values being possible.
Using this model, we can find the mean and variance and make predictions on the size of worms in the future.
But first we have to find the value of k.

A sketch of f(x) would look like:




With a **discrete** distribution, the sum of the probabilities was equal to 1.

The equivalent of this in a **continuous** distribution is that the area under the graph = 1

$$\int_0^6 kx^2\,dx = 1 \qquad k\left[\frac{x^3}{3}\right]_0^6 = k\,\frac{6^3}{3} = 1 \qquad \therefore\ k = \frac{1}{72}$$

$$\sum p(x) = \mathbf{1}$$

$$\int f(x)\,dx = \mathbf{1}$$

**Now we can ask some questions about our worms:** What is the probability that

a) A worm is less than 2 cm long $\quad P(X<2) = \int_0^2 \frac{1}{72}x^2\,dx$

b) A worm is greater than 5 cm long $\quad P(X>5) = \int_5^6 \frac{1}{72}x^2\,dx$ or $1 - \int_0^5 \frac{1}{72}x^2\,dx$

c) A worm is between 3 and 4 cm long $\quad P(3<X<4) = \int_3^4 \frac{1}{72}x^2\,dx$

$$P(X<a) = \int_0^a \frac{1}{72}x^2\,dx$$

$$P(X>a) = 1 - \int_0^a \frac{1}{72}x^2\,dx$$

**Note that point probabilities cannot be found**. For instance, if we wanted to find the probability of a worm being **exactly** 4cm long, we can't substitute x = 4 into f(x) and get $\frac{16}{72}$. The area under the graph equals 1 and the best we could do is to integrate between x = 3.9 and x = 4.1.

## THE CUMULATIVE DISTRIBUTION FUNCTION

We could do questions a, b and c above and many others concerning the lengths of worms by using the cumulative probability function. This would represent the "area to the left of **x**" and be denoted by F(x).

$$F(x) = \int_0^x \frac{1}{72}x^2\,dx = \frac{1}{72}\left[\frac{x^3}{3}\right]_0^x = \frac{x^3}{216}$$

a) P(worm is less than 2 cm long) $= P(X<2) = F(2) = \dfrac{8}{216}$

b) A worm is greater than 5 cm long $P(X>5) = 1 - F(5) = 1 - \dfrac{125}{216} = \dfrac{91}{216}$

c) A worm is between 3 and 4 cm long $P(3<X<4) = F(4) - F(3) = \dfrac{64}{216} - \dfrac{27}{216} = \dfrac{37}{216}$

**And as a check:** $\mathbf{F(6)} = \dfrac{216}{216} = \mathbf{1}$

**THE MEAN** - For a discrete PDF, we multiply each x-value by its probability and sum across all the values:

**For a continuous distribution: we multiply x by the probability function and integrate over the range……….**

$$\mathbf{E(X)} = \int_0^6 x\frac{x^2}{72}\,dx = \frac{1}{72}\left[\frac{x^4}{4}\right]_0^6 = \frac{1296}{288} = \mathbf{4.5}$$

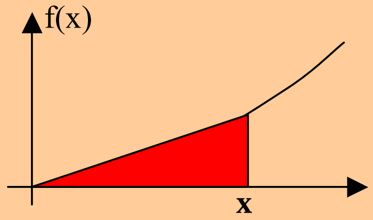
**the mean length of worm in my garden is 4.5cm.**

**Similarly, the expected value of $x^2$ is obtained by multiplying $x^2$ by the probability function and integrating……**

And the **VARIANCE** can be found using $\mathbf{E(X^2) - [E(X)]^2}$

The **MEDIAN** is the value m such that $P(X<m) = 0.5$, $F(m) = 0.5$, $\dfrac{m^3}{216} = 0.5$, $m^3 = 108$, $m = 4.8$

The **MODE** is the value of x which gives the greatest value of f(x) and may be found by differentiation: $f'(m) = 0$




**F(x)**

$$\mathbf{E(X) = \mu =} \sum xp(x)$$

$$\mathbf{E(X) = \mu =} \int_a^b xf(x)\,dx$$

$$\mathbf{E(X^2)} = \int_a^b x^2 f(x)\,dx$$

$$\mathbf{Var(X) = E(X^2) - [E(X)]^2}$$

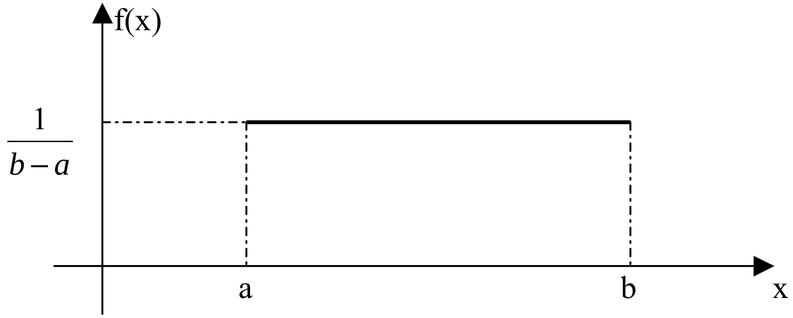
**Median:** $\int_a^m f(x)\,dx = \mathbf{0.5}$

**Mode: f '(m) = 0**

## THE CONTINUOUS UNIFORM DISTRIBUTION.

All probabilities are equally likely over a range a………b. The area under a rectangle must be 1 so the height must be $\frac{1}{b-a}$

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{For } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$




$E(X) = \frac{a+b}{2}$ (half way between a and b)

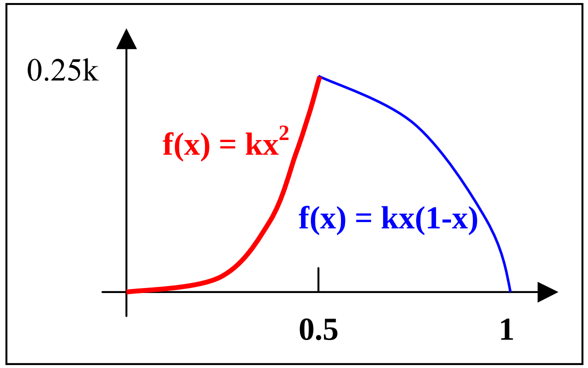
$Var(X) = \frac{(b-a)^2}{12}$

## SPLIT FUNCTIONS

The area under the graph over the entire range will still be **1** but the graph may come from two ore more functions, each with an associated range.

$$F(x) = \begin{cases} Kx^2 & 0 \le x \le 0.5 \\ kx(1-x) & 0.5 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

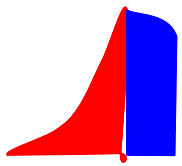
It is a good idea to sketch the function first even though we are usually asked to sketch the function in part (b) of a question. Base your sketch on your knowledge of curves like $y = kx^2$ and $y = kx(1-x)$ which is a quadratic meeting the x- axis at (0, 0) and (1, 0)




**The mode is of course 0.5.**

The area under the graph is equal to 1: $\int_0^{0.5} kx^2\,dx \quad + \quad \int_{0.5}^{1} k(x - x^2)\,dx \quad = \quad 1$

$k\frac{0.125}{3} + k\frac{1}{2} - k\frac{1}{3} - \left(k\frac{0.25}{2} - k\frac{0.125}{3}\right) = 1, \qquad k\frac{1}{6} = 1, \qquad \underline{\mathbf{k = 6}}$



$$P(X<0.6) = \int_0^{o.5} kx^2 dx + \int_{0.5}^{0.6} k(x - x^2) dx =$$