ALGEBRAIC PROOFS

Fermat's Last Theorem states that $x^n + y^n = z^n$ has no non-zero integer solutions for x, y and z when $n > 2$. Fermat wrote, *"I have discovered a truly remarkable proof which this margin is too small to contain"*.

There are many interesting relationships between different forms of numbers and there is a good chance that the exam paper will ask you to prove one of these.
These relationships exist between odd, even, square, triangular and even prime numbers.
You won't get round to proving Fermat's last theorem concerning primes since it has been proved by Andrew Wiles but you will get the chance to prove some others.
In a mathematical proof you have a line of reasoning consisting of many steps, that are almost self-evident. If the proof we write down is really rigorous, then nobody can ever prove it wrong. There are proofs that date back to the Greeks that are still valid today.

So let us start with some basics:
An expression for an even number is **2n.** – try substituting any integer value for n into **2n** and you will get an even number.
Adding 1 to an even number gives us an **odd number 2n + 1**
A **square number** will be represented by $\mathbf{n^2}$.
You also need to be aware about the products of even/odd numbers ………

odd x odd = odd
odd x even = even
even x even = even

1. Prove that the sum of two consecutive multiples of 5 is always an odd number.

Solution	***notes***
Let the consecutive multiples of 5 be 5n and 5(n + 1)	***notice that we change n to n + 1***
The sum is 5n + 5n + 5 = 10n + 5 = 5(2n + 1) which is odd.	***odd* x *odd* = *odd***

2. Prove that the product of two consecutive multiples of 5 is always an even number.

Let the two consecutive multiples of 5 be 5n and 5n + 5
The product is $5n(5n + 5) = 25n^2 + 25n = 25n(n + 1)$ which is even ***odd* x *even* x *odd***
or ***odd* x *odd* x *even***

3. Prove that $(n + 1)^2 - (n - 1)^2$ is a multiple of 4 for all positive integer values of n.

$(n + 1)^2 - (n - 1)^2 = n^2 + 2n + 1 - (n^2 - 2n + 1) = 4n$ which is a multiple of 4
Any expression which can be written with 4 as a factor is a multiple of 4. These include
4n, **4**n^2 **, 4**(3n = 1), **4**(n^2 + 2n + 1)… So if you can factorise **4** from any **expression**, it is a multiple of **4**

4. Prove that the sum of two consecutive multiples of 5 is always an odd number.

Let the two consecutive multiples of 5 be 5n and 5n + 5
The sum is 5n + (5n + 5) = 10n + 5 = 5(2n + 1) which is odd ***odd* x *odd* = *odd***

5. Prove that the sum of the squares of two consecutive even integers is never a multiple of 8.

Let the two consecutive even integers be 2n and 2n +2.
The sum of the squares is $(2n)^2 + (2n + 2)^2 = 4n^2 + 4n^2 + 8n + 4 = 8(n^2 + n) + 4$
which is not an exact multiple of 8. ***It is a multiple of 8 with remainder of 4.***

6. Prove that the sum of the squares of any 2 odd numbers leaves a remainder of 2when divided by 4.

Let the two odd numbers be $2m + 1$ and $2n + 1$ and the sum of their squares $(2m + 1)^2 + (2n + 1)^2$
$= 4m^2 + 4m + 1 + 4n^2 + 4n + 1 = 4(m^2 + n^2 + m + n) + 2$
Which leaves a remainder of 2 when divided by 4.

An expression of the form **p(function) + q** ***leaves a remainder of q when divided by p.***
This almost like completing the square but could be called completing the factor.
Learn this technique. Factorise part of the expression and leave the rest.

7. Investigate the square of the mean and the mean of the squares of 5 consecutive numbers.

Let the 5 consecutive numbers be n, n+1, n+2, n+3, n+4 and their mean $(5n + 10)/5 = n + 2$
The square of the mean is $(n + 2)^2 \quad = n^2 + 4n + 4$
The mean of the squares of the numbers is

$$(n^2 + n^2 + 2n + 1 + n^2 + 4n + 4 + n^2 + 6n + 9 + n^2 + 8n + 16)/5$$
$$= (5n^2 + 20n + 30)/5 \qquad = n^2 + 4n + 6$$

What can we say about the square of the mean and the mean of the squares of 5 consecutive numbers? Answer at the bottom of the page.

8. Prove that the difference between the squares of any 2 odd numbers is a multiple of 8.

Let the two odd numbers be $2m + 1$ and $2n + 1$ with m being bigger than n.
The difference between the squares is $(2m + 1)^2 - (2n + 1)^2 = 4m^2 + 4m + 1 - 4n^2 - 4n - 1$
$= 4(m^2 - n^2 + m - n)$ ***Which is certainly a multiple of 4***
But since $4(m^2 - n^2 + m - n)$ can be written $4[(m+n)(m-n) + (m-n)] = 4(m-n)(m + n + 1)$
A table will help to answer this one:

m	n	m-n	m + n + 1	(m-n)(m + n + 1)
Even	*Even*	*Even*	*Odd*	*Even*
Even	*Odd*	*Odd*	*Even*	*Even*
Odd	*Odd*	*Even*	*Odd*	*Even*
Odd	*Even*	*Odd*	*Even*	*Even*

The table shows that the product $(m-n)(m + n + 1)$ is always even and hence a multiple of 2.
So $4(m-n)(m + n + 1)$ is a multiple of 8.

9. Investigate the mean of consecutive integers.

Two consecutive integers like 6 and 7 have a mean of 6.5 which is not an integer.
Three consecutive integers like 6, 7 and 8 have a mean of 7. This is also the median.
So we can say that an <u>odd</u> number of consecutive integers will have a mean which is an integer.
The mean of the squares of 5 consecutive numbers is 2 more than the square of the mean.
Is there any relationship between the mean of the squares and the square of the mean of 3 consecutive numbers?
The table above states that an Even x Odd = Even. A quickie proof is:
$2n \times (2n + 1) = 4n^2 + 2n = 2(2n^2 + n)$ which is even.
Summary
An expression of the form 2(function) is always even
An expression of the form 2(function) + 1 is always odd
An expression of the form p(function) is always a multiple of p
An expression of the form p(function) + q leaves a remainder of q when divided by p.