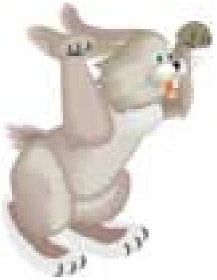
ARITHMETIC AND GEOMETRIC PROGRESSIONS

Arithmetic progressions



A rabbit hops along in bunny-units according to the following pattern:

3 7 11 15 19 23 27 ……… the first **T**erm of this sequence can be denoted by T_1 but we will **u**se U_1.

U_1 is equal to 3, U_2 is equal to 7 (3 + 4), U_3 is equal to 11 (3 + **2**x4) and U_7 is equal to 27 (3 + **6**x4)

The **seventh** term is equal to first term + **six** x difference

With first term denoted by **a**
And difference denoted by **d**

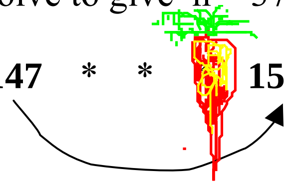
The formula for the nth term of the sequence is $U_n = a + (n-1)\,d$ **Learn!**

The 20^{th} term is 3 + 19x4 = 79. After 20 jumps the rabbit will be 79 bunny-units down the road.

If a juicy carrot were placed in its path 150 units down the road will he land on the carrot or hop over it?

Is 150 a member of the sequence? Assume it is and put $3 + (n-1)4 = 150$ Solve to give $n = 37\frac{3}{4}$

This is not a whole number. The $37\frac{3}{4}$ term is 150. i.e. 37^{th} term < 150, 38^{th} term > 150 ..**147** * * **151**.



The rabbit will hop over the carrot!

After how many jumps has the rabbit covered a distance of over 200 units? What is the first term to exceed 200?
Put $3 + (n-1)4 > 200$, $4n - 1 > 200$, $4n > 201$, $n > 50.25$. After **51** jumps.

Some problems will involve simultaneous equations:

If the tenth term of a sequence is 43 and the twentieth is 83 find the first term and the common difference.

$U_{10} = 43 = a + 9d$ (i)
$U_{20} = 83 = a + 19d$ (ii) subtract (i) from (ii) $40 = 10d$, **$d = 4$** and {substituting in (i)} **$a = 7$**.

Now suppose the sequence 3 7 11 15 19 23 27 ………

represents the amount of money, in pounds, you put into your piggy bank each w eek.

In the 10^{th} week: $U_{10} = 3 + 9x4 = 39$ You put £39 in the bank in the tenth week.

But how much do you have if we add up all the amounts after 10 weeks? We are looking for the sum to 10 terms:

Simply add up all the first 10 terms. Or write: $3 + 7 + 11 + \ldots\ldots 39 = S_{10}$ the **S**um to 10 terms.

Rewrite the terms in reverse: $39 + 35 + 31 + \ldots\ldots + 1 = S_{10}$

Add the two rows and notice: $42 + 42 + 42 + \ldots\ldots + 42 = 2S_{10}$

There are 10 lots of 42: $10 \times 42 = 2S_{10}$ therefore $S_{10} = \dfrac{10x42}{2} = 210$

Since **42** is the sum of the first term (a) and the last term (*l*) we have used the formula: …………… $\mathbf{S_n = \frac{n}{2}(a + l)}$

And since the last term *l*, may be written $l = a + (n - 1)d$, $S_n = \frac{n}{2}(a + a + (n - 1)d)$ …………… $\mathbf{S_n = \frac{n}{2}(2a + (n - 1)d)}$

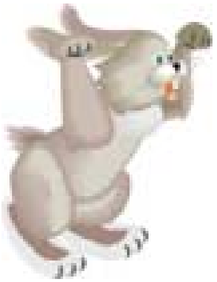
So after 20 weeks: $S_{20} = \frac{20}{2}(6 + 19x4) = 820$ you would have a total of £820.

After how many weeks will you have £990? Put $990 = \frac{n}{2}(6 + (n - 1)4)$ this leads to a Quadratic equation:

$1980 = n(6 + (n - 1)4)$, $4n^2 + 2n - 1980 = 0$, $2n^2 + n - 990 = 0$, $(2n + 45)(n - 22) = 0$ <u>$n = 22$</u>

When will the sum exceed £2000? Put $\frac{n}{2}(6 + (n - 1)4) > 2000$ and solve the resulting Quadratic.

Geometric progressions



A colony of rabbits grows according to the following pattern, which shows the number of rabbits after each year.

3 6 12 24 48 96   192 … this sequence does not have a common difference but a common multiplier or **r**atio.

U_1 is equal to 3, U_2 is equal to 3x2, U_3 is equal to 3x**2**x**2** or 3×2^{2}, U_7 is equal to 3×2^{6}.

The **seventh** term is equal to first term x **ratio** to the power of six

With first term denoted by **a**
And ratio denoted by **r**

The formula for the nth term of the sequence is $U_n = a\,r^{(n-1)}$ **Learn!**

The 20th term is $3 \times 2^{19} = 1572864$. After 20 years the rabbit colony would have grown to 1572864.

Is 1536 a member of the sequence? Assume it is and put $3 \times 2^{(n-1)} = 1536$ Solve (using logs?) to give **<u>n = 10</u>**

After how many years does the colony exceed 2000? Put $3 \times 2^{(n-1)} > 2000$ Solve to give $n > 10.38$ **<u>11 years</u>**

<u>Some problems will involve simultaneous equations:</u>

If the fifth term of a geometric series is 405 and the tenth is 98415, find the first term and the ratio.

$U_5 = 405 = ar^4$ (i)

$U_{10} = 98415 = ar^9$ (ii)   substitute (i) into (ii): $98415 = 405r^5$ $r^5 = 243$ **<u>r = 3</u>**

Now suppose the sequence 3 6 12 24 48 96 192 … represents the amount of money you put in your bank every month.

In the 10^{th} month: $U_{10} = 3x2^9 = 1536$ You put £1536 in the bank in the tenth month.

But how much do you have if we add up all the amounts after 10 months? We are looking for the sum to 10 terms:

Simply add up all the first 10 terms. Or write: $3 + 6 + 12 + \ldots\ldots 1536 = S_{10}$ the **S**um to 10 terms.
Multiply by r – in this case **2**: $2x3 + 2x6 + 2x12 + \ldots\ldots + 2x1536 = 2 \times S_{10}$
Subtract the two rows and notice: $3 - 2x1536 = S_{10} - 2S_{10}$
Replacing with **a** and **r**: $a - ar^n = S_n - r\,S_n$

$$a(1 - r^n) = S_n(1 - r)$$

The formula for the sum to n terms of a GP is

$$S_n = \frac{a(1-r^n)}{1-r} \quad \textbf{learn!}$$

Now suppose that a frog is sitting in the middle of a pond of radius 1 metre.

The frog hops to the edge in leaps of $\frac{1}{2}$ metre, then $\frac{1}{4}$ metre, then $\frac{1}{8}$ metre etc. since he is getting tired.

The sequence will be $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\ldots$ where $a = \frac{1}{2}$ and $r = \frac{1}{2}$. After 10 hops: $S_{10} = \frac{\frac{1}{2}(1-\frac{1}{2}^{10})}{1-\frac{1}{2}} = 0.999$m.

In cases where we have a GP with r, very small $\mathbf{-1 < r < 1}$, $\mathbf{r^n}$ gets smaller and smaller as n increases:

We say that that, as $n \longrightarrow \infty$, $\mathbf{r^n} \longrightarrow \mathbf{0}$ **" as n tends to infinity, r^n tends to zero".**

The formula $S_n = \frac{a(1-r^n)}{1-r}$ will then become $S_\infty = \frac{a}{1-r}$ The sum to infinity $= \frac{a}{1-r}$

After an infinite number of hops, the sum of the hops will be $\frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$. The frog will never get to the edge! Poor Froggy!