

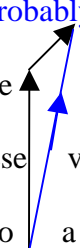
# VECTORS



Back in the good old days when on-board computers hadn't been invented, the co-pilot would have to work out the speed, direction and expected time of arrival of the plane.

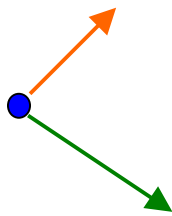
The Pilot would say, "We are flying north at 120 miles per hour" and the co-pilot would represent this by a scale drawing: ↑  
A straight line pointing north.

The co-pilot would then hear that the wind is blowing from the southwest at 30 miles per hour. He would then put this information on his diagram and calculate the resultant speed and direction of the plane. He would probably use the cosine formula for the calculation.

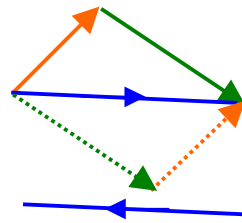
Note that the  effect of the wind coming from behind will make the plane go faster.

Because these velocities have a value with the direction we say **velocity is a vector** since it has magnitude and direction.

**Force** is also a vector: If two people are pulling an object with different forces in different directions, we could get rid of them and replace them with one person to have the same effect.



The forces have been drawn to scale. Position them nose to tail on another diagram and the resultant force will be clear. Or complete a parallelogram.



**One force** is equivalent to the other two. The resultant force taken in the opposite direction is the force, which would keep the object from moving at all.

If a quantity is not a vector it is said to be a scalar.

## COLUMN VECTORS

We have used column vectors before when we have studied transformations.

If the point A(2,3) is translated to the point B(5,7) we write

$$\vec{AB} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

The arrow above AB indicates a vector. From A to B we translate 3 units right and 4 units up.

Column vectors may be added, subtracted or multiplied by a scalar.

If  $\mathbf{a} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$  and  $\mathbf{b} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$   $\mathbf{a}$  and  $\mathbf{b}$  have been typed in **bold** to indicate vectors.

If **you** are writing the solution to a vector problem you will have to distinguish your vectors from scalars by using underlining.  $\underline{\mathbf{a}} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$  and  $\underline{\mathbf{b}} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$  I am going to use **bold** from now on.

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} + \begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad 3\mathbf{a} = 3 \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 6 \\ -12 \end{pmatrix}$$

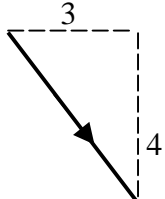
this vector is parallel to  $\mathbf{a}$  and 3 times greater in magnitude.

Find (i)  $3\mathbf{a} - 2\mathbf{b}$ , (ii)  $2(\mathbf{a} - \mathbf{b})$ , (iii) a vector parallel to  $\mathbf{a}$  (any scalar multiple of  $\mathbf{a}$  will be parallel to  $\mathbf{a}$ ).

Solve the equation  $\mathbf{a} + \mathbf{x} = \mathbf{b}$  (rearrange as in ordinary algebra:  $\mathbf{x} = \mathbf{b} - \mathbf{a}$ ).

**Vectors** such as **velocity** and **force** are represented by a line segment. The length of the line segment is the **magnitude** of the vector. The notation for the magnitude of a is  $|\mathbf{a}|$  and it is simply found using Pythagoras.

$\mathbf{a} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$



$|\mathbf{a}| = \sqrt{3^2 + 4^2} = 5.$

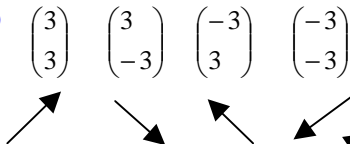
A square number is always positive so we ignore the negative sign in  $-4$  and simply write it as  $4^2$ .

A vector written as  $\begin{pmatrix} x\text{-entry} \\ y\text{-entry} \end{pmatrix}$  will be parallel to the y axis if the x-entry is zero  $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$  ↓

And parallel to the x axis if the y-entry is zero  $\begin{pmatrix} 3 \\ 0 \end{pmatrix} \rightarrow$  3 along zero up.

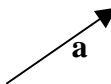
A vector with the x-entry numerically equal to the y-entry will be at  $45^\circ$

$\begin{pmatrix} 3 \\ 3 \end{pmatrix}$     $\begin{pmatrix} 3 \\ -3 \end{pmatrix}$     $\begin{pmatrix} -3 \\ 3 \end{pmatrix}$     $\begin{pmatrix} -3 \\ -3 \end{pmatrix}$

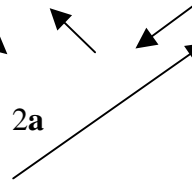


### WORKING WITH VECTOR ALGEBRA


Take a vector  $\mathbf{a}$



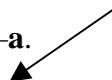
a vector parallel to  $\mathbf{a}$  and twice its magnitude is  $2\mathbf{a}$



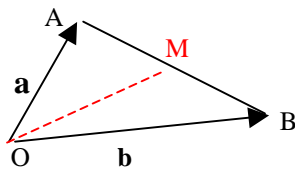
and with half its magnitude is  $\frac{1}{2}\mathbf{a}$



a vector with the same magnitude but in the opposite direction is  $-\mathbf{a}$ .



Taking the long route:

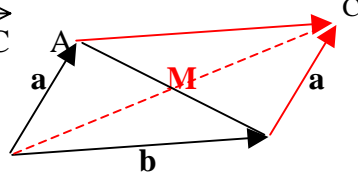


If  $\vec{OA} = \mathbf{a}$  and  $\vec{OB} = \mathbf{b}$  then  $\vec{AB} = -\mathbf{a} + \mathbf{b}$   
(to get from A to B go backward along  $\mathbf{a}$  and then along  $\mathbf{b}$ ).

Similarly,  $\vec{BA} = -\mathbf{b} + \mathbf{a}$

If M is the midpoint of AB, then  $\vec{OM} =$  (Taking the long route)  $\vec{OA} + \vec{AM} = \vec{OA} + \frac{1}{2}\vec{AB}$   
 $= \mathbf{a} + \frac{1}{2}(-\mathbf{a} + \mathbf{b}) = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$

If we draw the diagram again, we can see that  $\mathbf{a} + \mathbf{b}$  is the position vector  $\vec{OC}$



Therefore  $\frac{1}{2}(\mathbf{a} + \mathbf{b})$  is the position vector of the midpoint of  $\vec{OC}$ .

M is the midpoint of AB and also of OC.

So the diagonals of a parallelogram bisect one another.

There are many other geometrical results that can be proved using vector algebra and the rules above, with the additional rule: If **any point** K is taken on AB then since AK is a certain distance k along AB,

the vector  $\vec{OK} = \vec{OA} + k\vec{AB} = \mathbf{a} + k(-\mathbf{a} + \mathbf{b}) = (1-k)\mathbf{a} + k\mathbf{b}$

If we have a situation where  $m\mathbf{a} + n\mathbf{b} = p\mathbf{a} + q\mathbf{b}$  Then it must be true that  $m = p$  and  $n = q$ .

