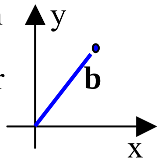
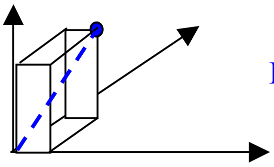
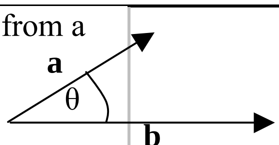
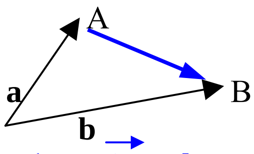
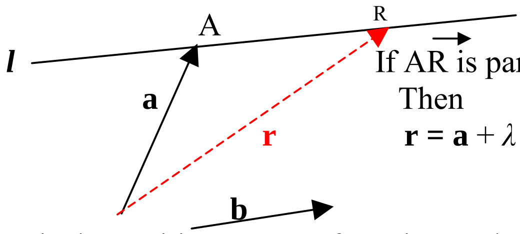
$\overrightarrow{\text{VECTORS}}$

Let us extend the GCSE work we have done on vectors in two dimensions to work with three dimensions.

TWO DIMENSIONS	THREE DIMENSIONS
If B is the point (3,4) and O is the origin We write $\overrightarrow{OB} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ as a column vector  This may also be written (3, 4)	If B is the point (2, 3, 4) and O is the origin $\overrightarrow{OB} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$  B(2, 3, 4) In 3 D it is like the corner of a wardrobe in a room.
MAGNITUDE OF A VECTOR	
The length of the line segment is the ***magnitude*** of the vector. The notation for the magnitude of **b** is **|b|** and it is simply found using Pythagoras. **|b|** = $\sqrt{3^2 + 4^2}$ = **5.**	Use a triple form of Pythagoras **|b|** = $\sqrt{2^2 + 3^2 + 4^2}$ = $\sqrt{29}$
UNIT VECTORS	
We define ***i*** as the unit vector in the **x** direction And ***j*** as the unit vector in the **y** direction. (these vectors appear in **bold** type in print but must be underlined in <u>your</u> solutions. You write <u>*i*</u> and <u>*j*</u> The vector **b** may be written in unit vector form: **b** = 3***i*** + 4***j*** The length of the line segment OB is 5 but to get a line of **unit** length we need to **divide by 5**. $(\frac{3}{5}, \frac{4}{5})$ The **unit vector** in the direction of $\overrightarrow{OB}$ is $\frac{3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{j}$	Here, as well as unit vectors in the x and y direction, we define ***k*** as the unit vector in the **z** direction. **b** = 2***i*** + 3***j*** + 4***k*** The length of the line segment OB is $\sqrt{29}$ The **unit vector** in the direction of $\overrightarrow{OB}$ is $\frac{3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{j} + \frac{4}{5}\boldsymbol{k}$
THE ANGLE BETWEEN TWO VECTORS	
There is a connection between two vectors from a point and the angle between them.  **a . b** = **|a| |b|** cos θ cos θ = $\frac{\textbf{a . b}}{\textbf{\|a\| \|b\|}}$ If **a** = a***i*** + b***j*** and **b** = c***i*** + d***j***, **a.b** = ac + bd In particular, if θ = 90°, cos 90 = 0 **a . b = 0**	This is known as the **scalar** product or the **dot** product. The answer is a **scalar.** If **a** = a***i*** + b***j*** + c***j*** and **b** = d***i*** + e***j*** + f***k***, **a.b** = ad + be + cf
THE DISTANCE BETWEEN TWO POINTS	
 If $\overrightarrow{OA}$ = a***i*** + b***j*** $\overrightarrow{OB}$ = c***i*** + d***j***, Then $\overrightarrow{AB}$ = **b - a** = (c-a)***i*** + (d-b)***j*** The distance AB is the modulus of (**b-a**)	If OA = 2***i*** + 3***j*** + 4***k*** If OB = 4***i*** + 2***j*** - 3***k*** $\overrightarrow{AB}$ = (4-2)***i*** + (2-3)***j*** + (-3-4)***k*** = 2***i*** -1***j*** -7***k*** The distance AB is $\sqrt{2^2 + 1^2 + 7^2}$ = $\sqrt{54}$

THE VECTOR EQUATION OF A LINE THROUGH A POINT AND PARALLEL TO A CERTAIN VECTOR

Take the point A with position vector **a** and the vector **b**.



If $\overrightarrow{AR}$ is parallel to **b**

Then $\overrightarrow{AR} = \lambda\,\mathbf{b}$

$\mathbf{r} = \mathbf{a} + \lambda\,\mathbf{b}$

a is the position vector of a point on the line and **b** is the direction vector of the line ***l***.
λ is a scalar which gives different positions for R up and down the line.

To find the vector equation of the line that passes through the point A with position vector $2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$ and is parallel to the vector $4\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$.

The equation of the line is

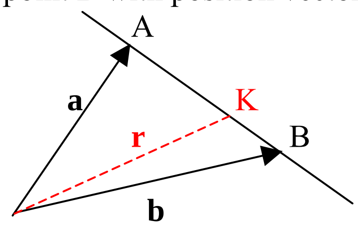
$\mathbf{r} = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k} + \lambda(4\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k})$

The equation of the line may be written:

$\mathbf{r} = (2+4\lambda)\boldsymbol{i} + (3+2\lambda)\boldsymbol{j} + (4-3\lambda)\boldsymbol{j}$

THE VECTOR EQUATION OF A LINE THROUGH TWO POINTS

Take the point A with position vector **a** and the point B with position vector **b**.



Let $\overrightarrow{AK} = \lambda\overrightarrow{AB}$

Then $\mathbf{r} = \mathbf{a} + \lambda\overrightarrow{AB}$

$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$

$\mathbf{r} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$

To find the vector equation of the line that passes through the points A and B with position vectors $2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$ and $4\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$ respectively.

The equation of the line is

$\mathbf{r} = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k} + \lambda\{(4-2)\boldsymbol{i} + (2-3)\boldsymbol{j} + (-3-4)\boldsymbol{k}\}$

$\mathbf{r} = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k} + \lambda\{2\boldsymbol{i} - 1\boldsymbol{j} - 7\boldsymbol{k}\}$

If two lines have vector equations: l_1: $\mathbf{r}_1 = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k} + \lambda\{2\boldsymbol{i} - 4\boldsymbol{j} - 7\boldsymbol{k}\}$

And l_2: $\mathbf{r}_2 = 3\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k} + \mu\{1\boldsymbol{i} - 1\boldsymbol{j} - 4\boldsymbol{k}\}$

We can compare coefficients: $2 + 2\lambda = 3 + \mu$, $3 - 4\lambda = -2 - \mu$, $4 - 7\lambda = 2 - 4\mu$

Solving simultaneously: $5 - 2\lambda = 1 \;\therefore\; \lambda = 2$ *and* $\mu = 3$

Confirm that this fits the third equation $4 - 7(2) = 2 - 4(3)$ and find **the point of intersection**

$\mathbf{r}_1 = 6\boldsymbol{i} - 5\boldsymbol{j} - 10\boldsymbol{k}$ $\mathbf{r}_2 = 6\boldsymbol{i} - 5\boldsymbol{j} - 10\boldsymbol{k}$ ***The point of intersection is (6, -5, -10)***

To find the **angle** between the two lines use the **dot product** with the **direction vectors** of both lines.

Ie. $2\boldsymbol{i} - 4\boldsymbol{j} - 7\boldsymbol{k}$ ***and*** $1\boldsymbol{i} - 1\boldsymbol{j} - 4\boldsymbol{k}$ $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{2 + 4 + 28}{\sqrt{69}\,\sqrt{18}} = 0.96$, $\theta = 15.26^\circ$

If the angle is 90° then the lines are **perpendicular**. Conversely, if $\mathbf{a}\cdot\mathbf{b} = 0$ then the angle is 90°

<u>WORKING WITH VECTOR ALGEBRA</u> – Revision – See the factsheet on GCSE vectors.